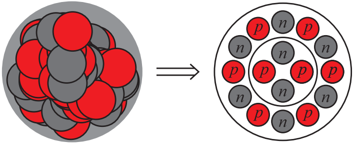
Nuclear structure theory II: Pairing, deformation, and collective nuclear structure

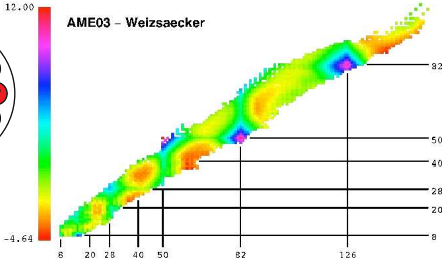
Mark A. Caprio
Department of Physics and Astronomy
University of Notre Dame

Exotic Beam Summer School
Notre Dame, IN
June 10, 2022

UNIVERSITY OF NOTRE DAME

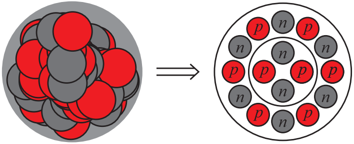
From simple shell structure to collective dynamics

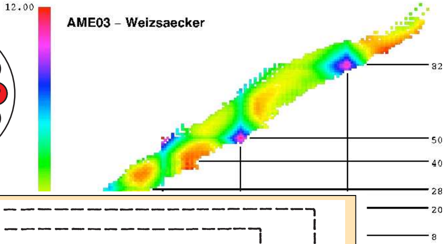


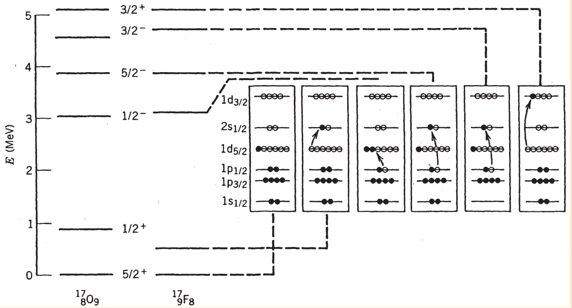


P. Van Isacker, AIP Conf. Proc. No. 819 (2006), p. 57.

From simple shell structure to collective dynamics

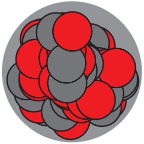


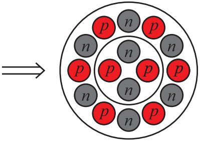


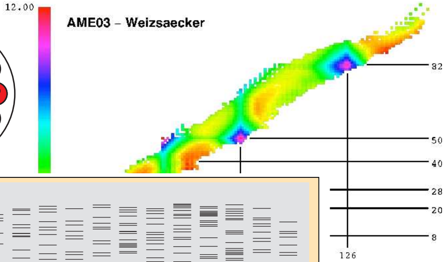


Kenneth S. Krane, *Introductory Nuclear Physics* (Wiley, New York, 1988).

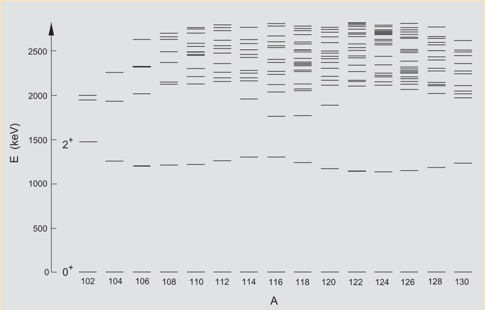
From simple shell structure to collective dynamics







p. 57.



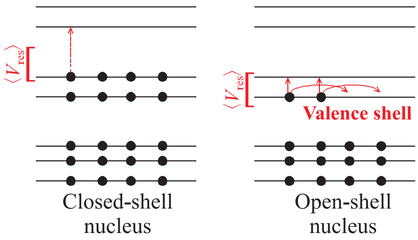
David J. Rowe and John L. Wood, *Fundamentals of Nuclear Models* (World Scientific, Singapore, 2010).

Shell model and collective correlations

$$H=\sum_{i=1}^{A}-\frac{\hbar^2}{2m_i}\nabla_i^2+\sum_{i,j=1}^{A}V(\mathrm{r}_i-\mathrm{r}_j)$$

$$\Rightarrow \overbrace{\sum_{i=1}^{A}\left[-\frac{\hbar^2}{2m_i}\nabla_i^2+U(r_i)\right]}^{H_0}$$

$$+\underbrace{\left[\sum_{i,j=1}^{A}V(\mathrm{r}_i-\mathrm{r}_j)-\sum_{i=1}^{A}U(r_i)\right]}_{V_{\mathrm{res}}}$$



Independent particle model ($H \approx H_0$): Eigenstate approximated as single configuration

Classic shell model ("configuration interaction" calculation):

Many-body problem restricted to *valence* shell

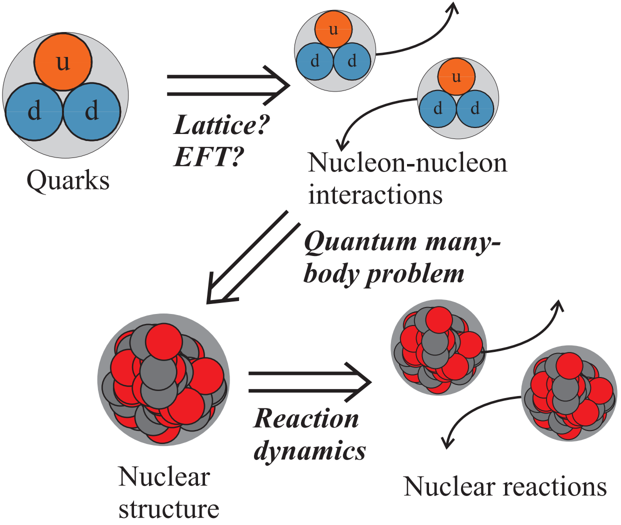
Neglected ("inert") core leads to effective interaction of valence nucleons

Open shell [$\Delta\varepsilon \lesssim \langle V_{\mathrm{res}}\rangle$] permits collective phenomena:

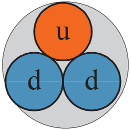
Large number of single-particle configurations energetically accessible

Little energy required for excitation

Flow chart for *ab initio* nuclear theory

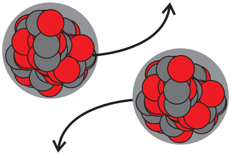


Flow chart for phenomenological nuclear theory (extreme case)



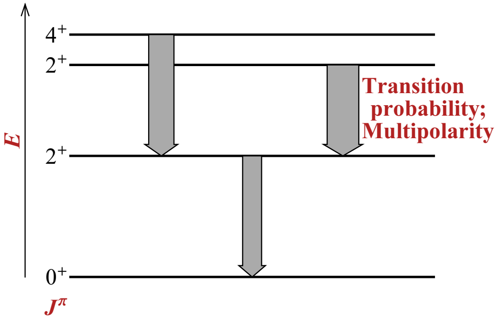
Quarks

...then a miracle occurs...



Nuclear reactions

(with apologies to S. Harris)



Obtain detailed information on physical structure and excitation phenomena from spectroscopic properties

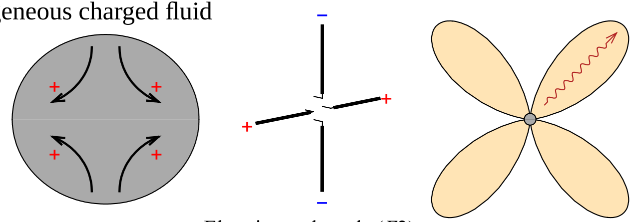
– Level energies and quantum numbers
– Electromagnetic transition probabilities and multipolarities

Fermi's golden rule $\quad T_{i\rightarrow f} \propto |\langle \Psi_f | \hat{T} | \Psi_i \rangle|^2$

Electromagnetic probes (e-scattering), α decay, β decay, nucleon transfer reactions, . . .

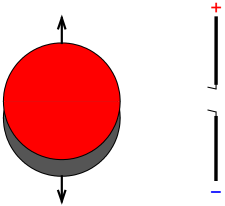
Multipolarity of nuclear radiation

Homogeneous charged fluid

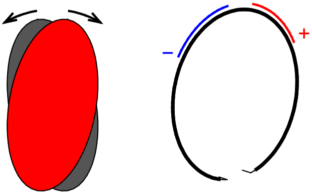


Electric quadrupole ($E2$)

Relative motion of proton and neutron density



Electric dipole ($E1$)



Magnetic dipole ($M1$)

Multipole operator definitions

Electric quadrupole ($E2$) operator

$$\mathbf{Q}_2 = \sum_{i=1}^{A} e_i r_i^2 Y_{2u}(\hat{\mathbf{r}}_i) \qquad = \mathbf{Q}_p + \mathbf{Q}_n$$

$$\boxed{e_p = e \quad e_n = 0}$$

Magnetic dipole ($M1$) operator

$$\mathbf{M}_1 = \sqrt{\frac{3}{4\pi}}\mu_N \sum_{i=1}^{A} (g_\ell^{(i)} \boldsymbol{\ell}_i + g_s^{(i)} s_i)$$

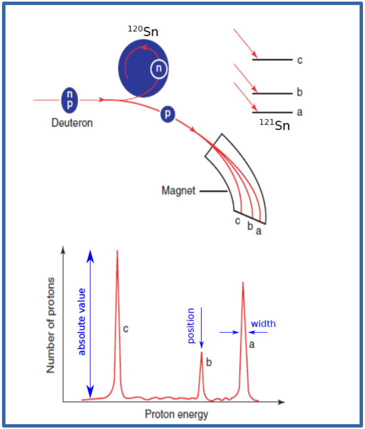
$g_{\ell,p} = 1$	$g_{\ell,n} = 0$
$g_{s,p} \approx 5.585$	$g_{s,n} \approx -3.826$

$$= g_{\ell,p}\mathbf{L}_p + g_{\ell,n}\mathbf{L}_n + g_{s,p}\mathbf{S}_p + g_{s,n}\mathbf{S}_n$$

$$\mathbf{L}_p = \sum_{i=1}^{Z} \boldsymbol{\ell}_{p,i} \quad \mathbf{L}_n = \sum_{i=1}^{N} \boldsymbol{\ell}_{n,i} \quad \mathbf{S}_p = \sum_{i=1}^{Z} \mathbf{s}_{p,i} \quad \mathbf{S}_n = \sum_{i=1}^{N} \mathbf{s}_{n,i}$$

Nuclear structure with transfer reactions

- transfer reactions probe nuclear response to the addition of nucleon
- information about nuclear structure from:
    - *angular differential cross section*
    - *absolute value*
    - *position*
    - *width (in the continuum)*



A standard approach to reactions:

$$\boxed{\sigma = S_i^2 \tilde{\sigma}}$$

spectroscopic factor from structure model

cross section from few-body/reaction models

can suffer from inconsistency between the two schemes !

courtesy of J. Rotureau and G. Potel

Outline

- Isospin
- Pairing
- Deformation: Rotation

Observed energy levels for $A = 6$ nuclei

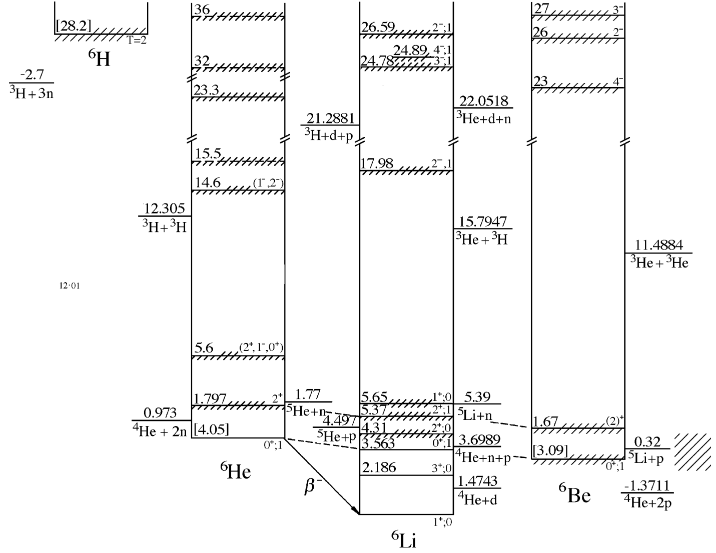
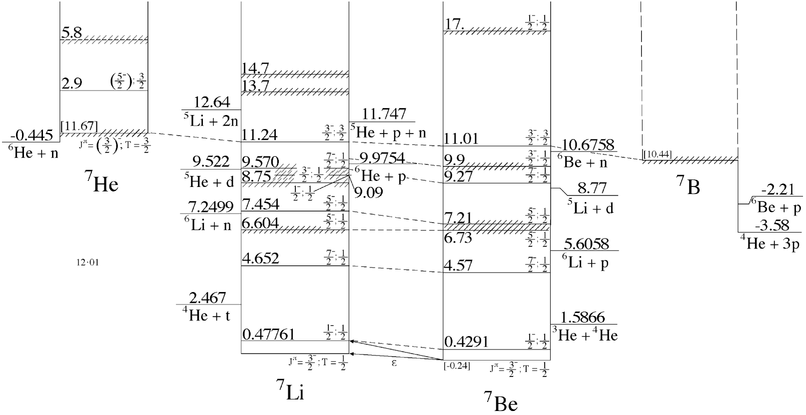


Figure from D.R. Tilley *et al.*, Nucl. Phys. A **708**, 3 (2002).

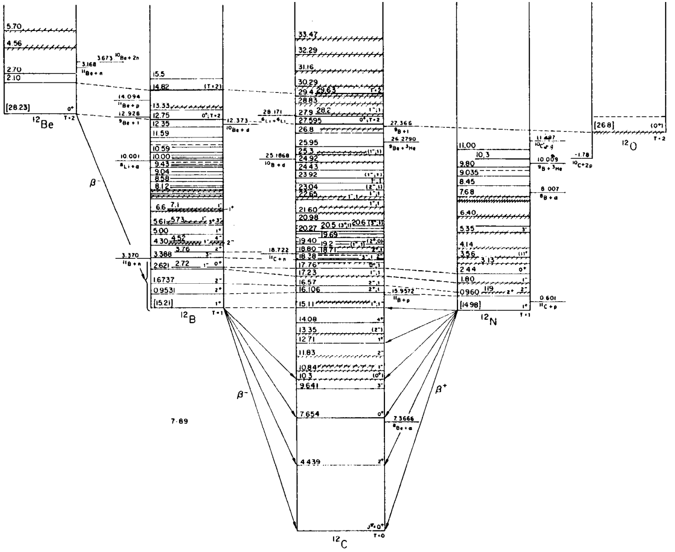
Observed energy levels for $A = 7$ nuclei



M. A. Caprio, University of Notre Dame

Figure from D.R. Tilley *et al.*, Nucl. Phys. A **708**, 3 (2002).

Observed energy levels for A = 12 nuclei



M. A. Caprio, University of Notre Dame

Figure from F. Ajzenberg-Selove, Nucl. Phys. A **506**, 1 (1990).

What is a Lie algebra?

A vector space...

$$\begin{array}{ll} X,Y \in \mathscr{V} \;\Rightarrow\; X+Y \in \mathscr{V},\; aX \in \mathscr{V} & \text{Closure} \\ a(X+Y) = aX + aY & \text{Linearity} \end{array}$$

...with a vector product ("Lie product")

$$\begin{array}{ll} [X,Y] \in \mathscr{V} & \text{Closure} \\ [aX+bY,Z] = a[X,Z] + b[Y,Z] & \text{Linearity} \\ [X,Y] = -[Y,X] & \text{Antisymmetry} \\ [A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0 & \text{Jacobi identity} \end{array}$$

Note: A vector space is spanned by d basis vectors X_1, X_2, $\ldots$, X_d.

$$\mathscr{V} = \left\{ \sum_{i=1}^{d} a_i X_i \;\middle|\; a_i \in \mathbb{R} \text{ or } \mathbb{C} \right\} \qquad d = \text{“dimension” of } \mathscr{V}$$

Why should we care?

There are Lie algebras hidden inside our QM problems! Two vector spaces...

1) Space of *states* — “Hilbert space”

2) Space of *operators* (!)

$$a(\hat{X}+\hat{Y}) = a\hat{X} + a\hat{Y}$$

$$[\hat{X},\hat{Y}] \equiv \hat{X}\hat{Y} - \hat{Y}\hat{X} \quad \text{is a “Lie product”} \checkmark$$

EXAMPLE Angular momentum algebra [SU(2)]

basis J_x, J_y, J_z

closure? $[J_x,J_y] = iJ_z \quad [J_y,J_z] = iJ_x \quad [J_z,J_x] = iJ_y \quad \checkmark$

Lie algebra operators as “generators” for continuous transformations

Lie algebra $g \overset{R=e^{iX}}{\Longleftrightarrow}$ Lie group G *e.g.*, $R(\theta) = e^{i(\theta_x J_x + \theta_y J_y + \theta_z J_z)}$

Symmetry — invariance of Hamiltonian under tranformation

$$R(\theta)HR(\theta)^\dagger = H \quad \Leftrightarrow \quad [J_i,H] = 0 \quad (i = 1,2,3)$$

Eigenvalues form degenerate multiplets $(M = -J,\ldots,J-1,J)$

Eigenstates rotate into each other $R(\theta)|JM\rangle = \sum_{M'=-J}^{J} \mathscr{D}^{(J)}_{M'M}(\theta)|JM'\rangle$

SU(2) in a nutshell

$\mathrm{SU}(2) \sim \mathrm{SO}(3)$ (2 × 2 spin rotation or 3 × 3 Euler rotation matrices)

Ladder operators	$J_\pm = J_x \pm iJ_y$	*Raises or lowers M value*
Weight operator	$J_0 = J_z$	*"Weighs" a state for its M value*

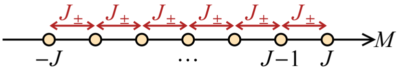
Everything follows from the commutators*...

$$[J_0, J_+] = +J_+ \quad [J_0, J_-] = -J_- \quad [J_+, J_-] = 2J_0$$

Action of generators

$$J_\pm |JM\rangle = \sqrt{(J \mp M)(J \pm M + 1)}\,|J(M\pm 1)\rangle \qquad J_0|JM\rangle = M|JM\rangle$$

States form "irreducible representation" connected by ladder operators



Set may be labeled by its "highest weight" M_{max} $(\equiv J)$

*Actually, these plus the relations $J_+^\dagger = J_-$ and $J_0^\dagger = J_0$...

Symmetries in nuclei

Fundamental symmetries

- Rotation [SU(2)] & parity $\Rightarrow$ J, P

Approximate symmetries of the many-body problem

- Isospin [SU(2)] & Wigner spin-isospin [SU(4)]
- Pairing quasispin symmetries: SU(2), SO(5), …
- Phase space (or oscillator) symmetries: Elliott SU(3) & $\mathrm{Sp}(3,\mathbb{R})$

Symmetries of collective degrees of freedom

- Bosonic models: U(6), …
- Symplectic collective model [$\mathrm{Sp}(3,\mathbb{R})$ *again!*] *Collective flow*

D. J. Rowe, A. E. McCoy, and M. A. Caprio, Physica Scripta **91**, 033003 (2016).

But symmetries are broken, so... Why symmetries?

- Identifying and characterizing emergent correlations

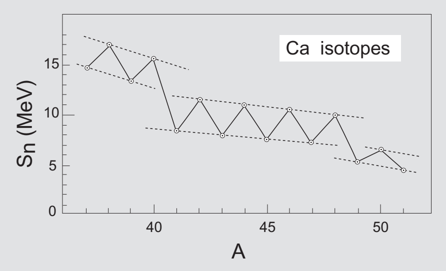
  E.g., isospin multiplets, Elliott rotation
- Symmetry as computational tool $H = H^{(0)}_{symm} + H'$

  "Right" basis for decomposing and truncating many-body space

Outline

- Isospin
- Pairing
- Deformation: Rotation

Figure 1.8: One-neutron separation energies, S_n, for the calcium isotopes. Note the odd-even staggering between neighbouring nuclei and the strong discontinuities that occur between $A = 40$ and 41 and between $A = 48$ and 49 (cf. Figure 1.9). (The data are from Audi G., Wapstra A.H. and Thibault C. (2003), *Nucl. Phys.* **A729**, 337.)



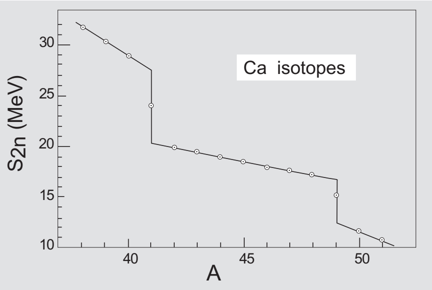
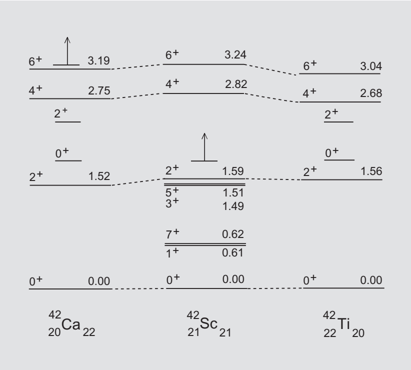


Figure 1.9: Two-neutron separation energies, S_{2n}, for the calcium isotopes. The odd-even staggering is smoothed away, leaving a clear indication of discontinuities at $A = 41$ and 49. (The data are from Audi G., Wapstra A.H. and Thibault C. (2003), *Nucl. Phys.* **A729**, 337.)

Single-particle energies in the *pf* shell

Figure courtesy United States Postal Service

Figure 1.28: Low-energy states in the $A = 42$ isobars ^{42}Ca, ^{42}Sc, and ^{42}Ti. Excitations are in MeV. Levels are labelled by their spin-parity, J^π. The vertical arrows indicate the energies above which there are excited states known but which are omitted from the figure. The states shown for ^{42}Sc result from the various spin couplings of the configuration $\pi 1f_{7/2}\nu 1f_{7/2}$. The $J = 0$, 2, 4, 6 members of this multiplet are connected with the corresponding $(\pi 1f_{7/2})^2$ and $(\nu 1f_{7/2})^2$ states in ^{42}Ti and ^{42}Ca, respectively. (The data are taken from Endt P.M. (1990), *Nucl. Phys.* **A521**, 1.)



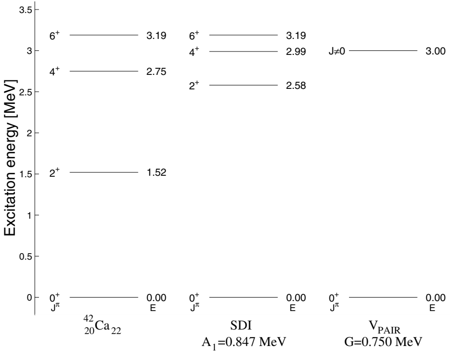
Pairing as approximation to short-range interaction

$$V_{\mathrm{RES}} = -\tfrac{1}{2} \sum_J \widehat{J} \langle j\,j\,;\,J\,|V|\,j\,j\,;\,J\rangle \Big[\big[c_j^\dagger c_j^\dagger\big]_J \big[\tilde{c}_j \tilde{c}_j\big]_J \Big]_{00}$$

$$\overset{\text{pairing}}{\approx} -\tfrac{1}{2} \langle j\,j\,;\,0\,|V|\,j\,j\,;\,0\rangle \big[c_j^\dagger c_j^\dagger\big]_0 \big[\tilde{c}_j \tilde{c}_j\big]_0$$

$$= -\tfrac{1}{2} \langle j\,j\,;\,0\,|V|\,j\,j\,;\,0\rangle \widehat{j}^{-2} \sum_{mm'} (-1)^{j-m+j-m'} c_{jm}^\dagger c_{j,-m}^\dagger \tilde{c}_{jm'} \tilde{c}_{j,-m'}$$

$$= \tfrac{1}{2} \langle j\,j\,;\,0\,|V|\,j\,j\,;\,0\rangle \widehat{j}^{-2} \sum_{mm'} c_{jm}^\dagger \tilde{c}_{jm}^\dagger \tilde{c}_{jm'} c_{jm'}$$



M. A. Caprio, University of Notre Dame

J. Suhonen, *From Nucleons to Nucleus* (Springer-Verlag, Berlin, 2007).

Seniority relates states obtained by adding pairs

$$\boxed{\begin{aligned} \left[A, A^\dagger\right] &= 1 - \hat{n}/\Omega \ , \\ \left[A^\dagger, \hat{n}\right] &= -2A^\dagger \ , \\ \left[V_{\mathrm{PAIR}}, A^\dagger\right] &= -GA^\dagger(\Omega - \hat{n}) = -G(\Omega - \hat{n} + 2)A^\dagger \ . \end{aligned}}$$

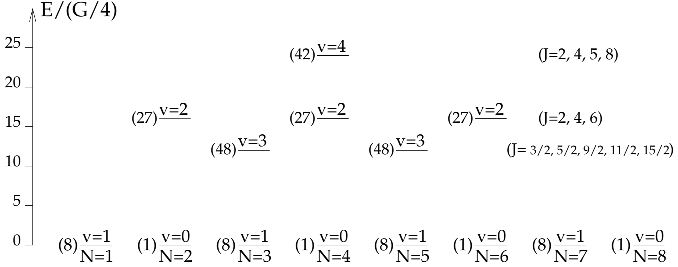


Fig. 12.3. Excitation spectra in the seniority scheme for different numbers N of particles occupying the $0\mathrm{f}_{7/2}$ shell. The seniority v is indicated for each level. The numbers in parentheses to the left of the levels give the degeneracies. The angular momentum content of the levels is given on the far right

J. Suhonen, *From Nucleons to Nucleus* (Springer-Verlag, Berlin, 2007).

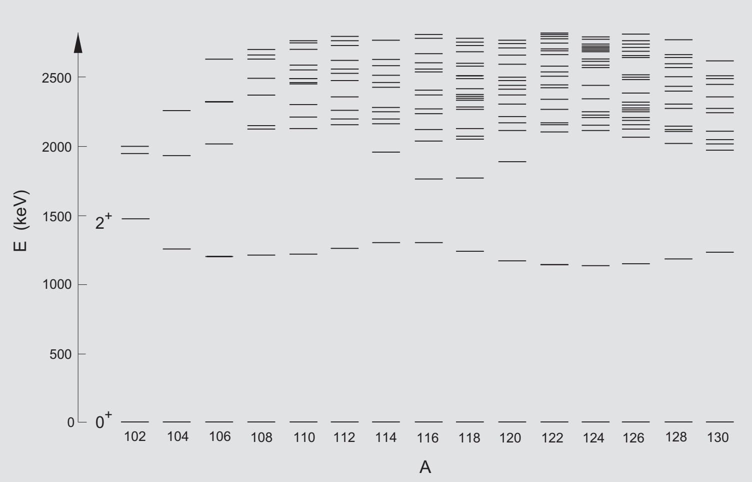


Figure 1.29: Low-energy states in the even-mass tin ($Z = 50$) isotopes. The 0^+ ground states and 2^+ first excited states are discussed in the text. (The data are taken from *Nuclear Data Sheets* and Juutinen S. *et al.* (1997), *Nucl. Phys.* **A617**, 74 – ^{106}Sn; Górska M. *et al.* (1998), *Phys. Rev.* **C58**, 108 – ^{104}Sn.)

D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

M. A. Caprio, University of Notre Dame

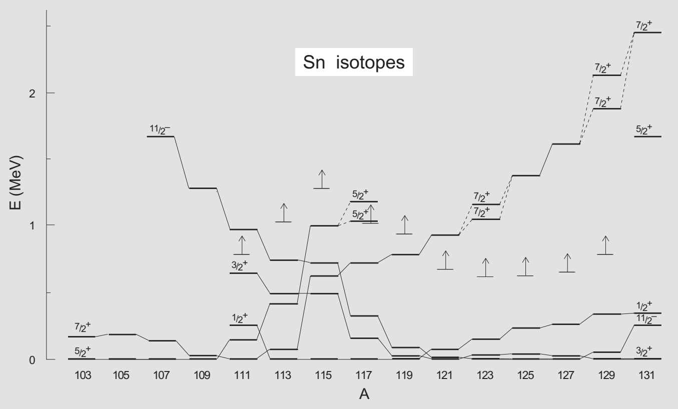
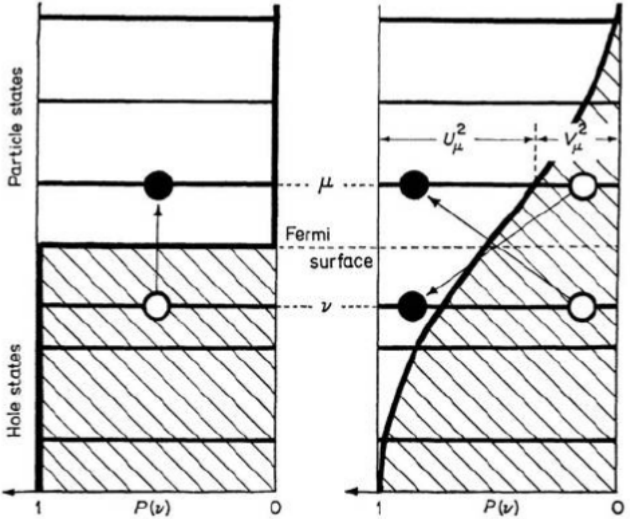


Figure 1.30: Low-energy states in the odd-mass tin isotopes. Levels are labelled by their spin-parity. The vertical arrows indicate the energies above which there are excited states known but which are omitted from the figure. The lowest three states are a selection from the spin-parities $5/2^+$, $7/2^+$, $1/2^+$, $3/2^+$, $11/2^-$, corresponding to the single-particle configurations $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$, $1h_{11/2}$, respectively. Information on states in ^{103,105,107,109}Sn is very limited. The identification of $2\mathrm{d}_{5/2}$ in ^{117}Sn and $1\mathrm{g}_{9/2}$ in ^{123,129}Sn is ambiguous. (The data are taken from *Nuclear Data Sheets* and Fahlander C. *et al.* (2001), *Phys. Rev.* **C63**, 021307(R) – ^{103}Sn.)

D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

M. A. Caprio, University of Notre Dame

The Fermi surface and quasiparticle excitation



D. J. Rowe, *Nuclear Collective Motion: Models and Theory* (World Scientific, Singapore, 2018).

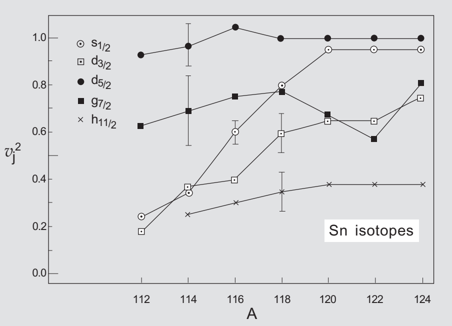


Figure 1.31: Fractional occupation probabilities, v_j^2, of single-particle orbitals in $^{112-124}$Sn. The uncertainties in v_j^2 shown are typical for each subshell (other uncertainties are omitted to avoid cluttering the figure). (The data are taken from Fleming D.G. (1982), *Can. J. Phys.* **60**, 428.)

D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

Outline

– Isospin
– Pairing
– Deformation: Rotation

Separation of rotational degree of freedom

Factorization of wave function $|\psi_{JKM}\rangle$ $\qquad J = K, K+1, \ldots$

$|\phi_K\rangle$ *Intrinsic structure* *($K \equiv$ a.m. projection on symmetry axis)*

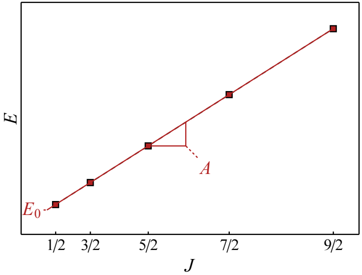
$\mathcal{D}^J_{MK}(\vartheta)$ *Rotational motion in Euler angles ϑ*

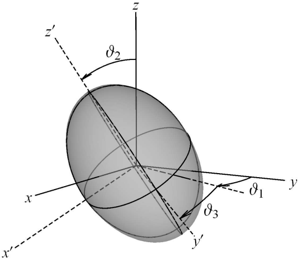
Rotational energy

$$E(J) = E_0 + A[J(J+1) + \underbrace{a(-)^{J+1/2}(J+\tfrac{1}{2})}_{\text{Coriolis } (K=1/2)}] \qquad A \equiv \frac{\hbar^2}{2\mathcal{J}}$$

Rotational relations (Alaga rules) on electromagnetic transitions

$$B(E2; J_i \rightarrow J_f) \propto (J_i K 2 0 | J_f K)^2 (eQ_0)^2 \qquad eQ_0 \propto \langle \phi_K | Q_{2,0} | \phi_K \rangle$$





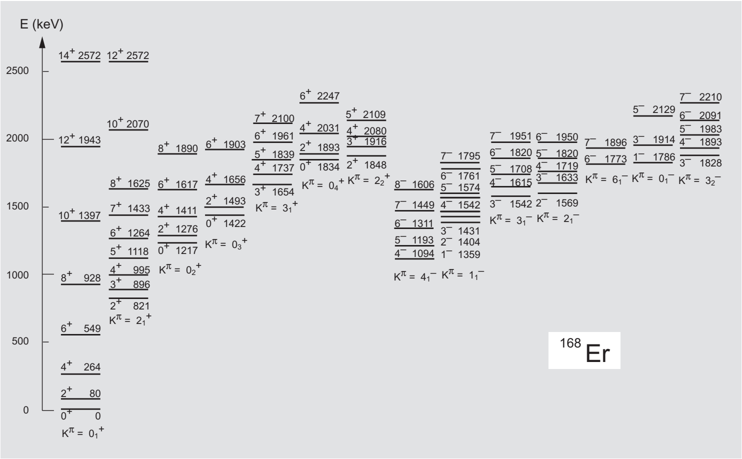


Figure 1.58: The low-lying states of ^{168}Er arranged into rotational bands. The positive-parity bands are shown also in Figure 1.59. (The data are taken from *Nuclear Data Sheets*.)

Figure from D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

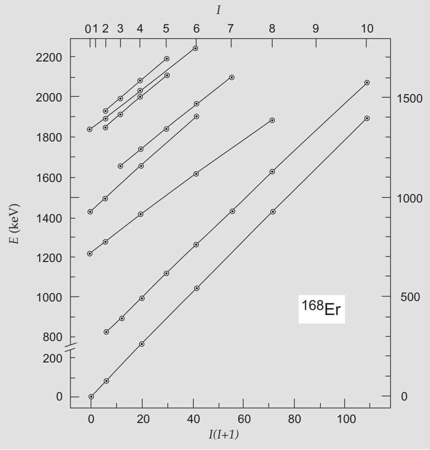


Figure 1.59: Plots of the excitation energies of the low-lying positive-parity states of ^{168}Er (cf. Figure 1.58) versus $I(I+1)$, cf. Equation (1.53) to reveal rotational behaviour. Note the left and right energy scales, with the left-hand scale applying to all but the ground-state band.

Figure from D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

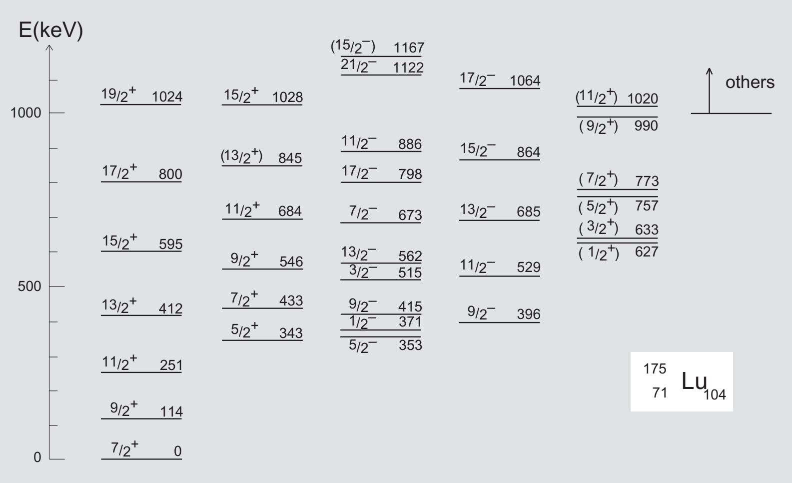


Figure from D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

Separation of rotational degree of freedom

Factorization of wave function $|\psi_{JKM}\rangle$ $\qquad J = K, K+1, \ldots$

$|\phi_K\rangle$ *Intrinsic structure* *($K \equiv$ a.m. projection on symmetry axis)*

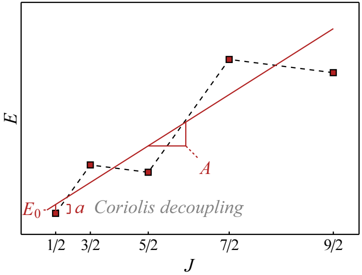
$\mathcal{D}^J_{MK}(\vartheta)$ *Rotational motion in Euler angles ϑ*

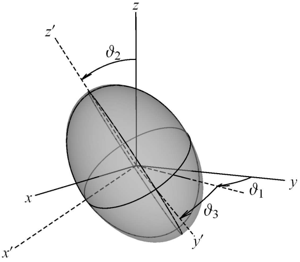
Rotational energy

$$E(J) = E_0 + A[J(J+1) + \overbrace{a(-)^{J+1/2}(J+\tfrac{1}{2})}^{\text{Coriolis } (K=1/2)}] \qquad A \equiv \frac{\hbar^2}{2\mathcal{J}}$$

Rotational relations (Alaga rules) on electromagnetic transitions

$$B(E2; J_i \rightarrow J_f) \propto (J_i K 20 | J_f K)^2 (eQ_0)^2 \qquad eQ_0 \propto \langle \phi_K | Q_{2,0} | \phi_K \rangle$$





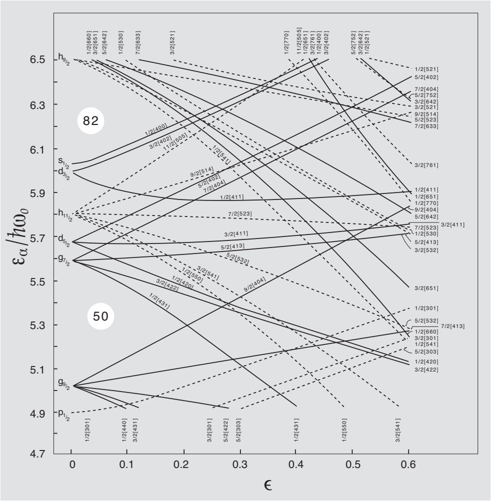


Figure 1.71: A Nilsson diagram for protons in nuclei with $50 \leq Z \leq 82$. Energies, in units of $\hbar\omega_0$, are plotted against the deformation parameter, ϵ. Energy levels are labelled by their spherical shell model quantum numbers, l and j, at $\epsilon = 0$, and by the asymptotic quantum numbers $\Omega[Nn_z\Lambda]$ for $\epsilon \neq 0$. (The figure is adapted from Lamm I.-L. (1969), *Nucl. Phys.* **A125**, 504.)

D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

Observed energy levels for $A = 7$ nuclei

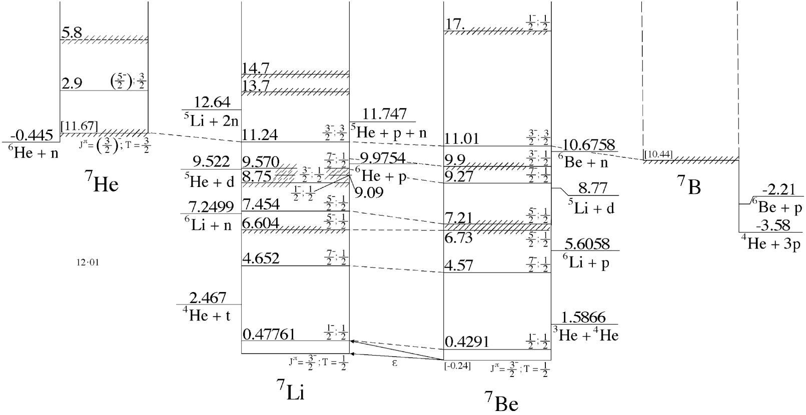
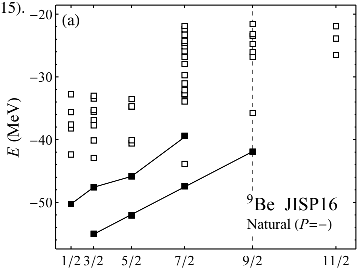


Figure from D.R. Tilley *et al.*, Nucl. Phys. A **708**, 3 (2002).

Rotational features emerge in *ab initio* calculations

P. Maris, M. A. Caprio, and J. P. Vary, Phys. Rev. C **91**, 014310 (2015).
C. W. Johnson, Phys. Rev. C **91**, 034313 (2015).
M. A. Caprio, P. J. Fasano, P. Maris, A. E. McCoy, J. P. Vary, Eur. Phys. J. A **56**, 120 (2020).



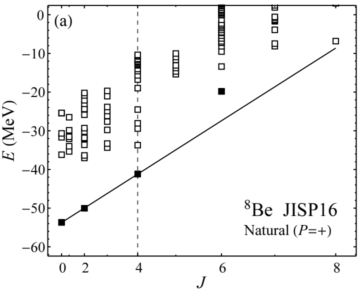
Valence shell structure? SU(3)

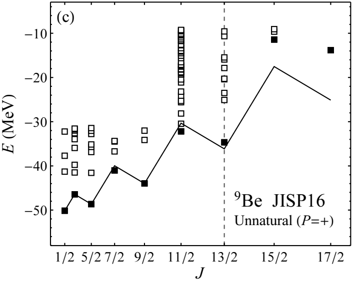
T. Dytrych *et al.*, Phys. Rev. Lett. **111**, 252501 (2013).

Multishell dynamics? $Sp(3,\mathbb{R})$

A. E. McCoy *et al.*, Phys. Rev. Lett. **125**, 102505 (2020).

Cluster rotation?





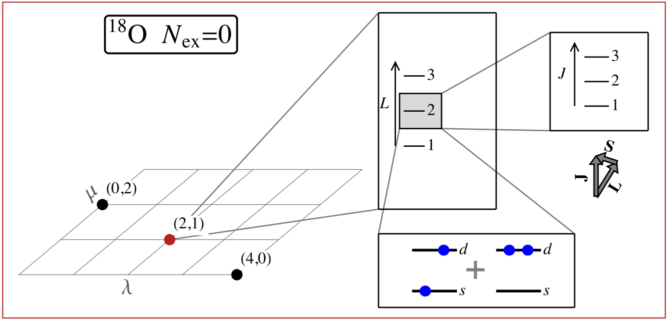
Elliott SU(3) symmetry

Generators of SU(3) $\supset$ SO(3)

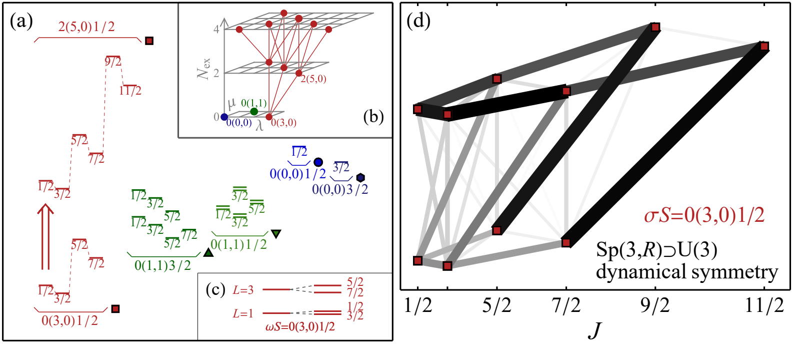
$$L_M^{(1)} \sim (b^\dagger \times \tilde{b})_M^{(1)} \qquad Q_M^{(2)} \sim (b^\dagger \times \tilde{b})_M^{(2)}$$

States classified into SU(3) irreps (λ,μ)

- States are correlated linear combinations of configurations over ℓ-orbitals
- Branching of SU(3) $\rightarrow$ SO(3) gives rotational bands (in L)



$\mathrm{Sp}(3,\mathbb{R}) \supset \mathrm{U}(3)$ dynamical symmetry for $^{7}\mathrm{Be}$



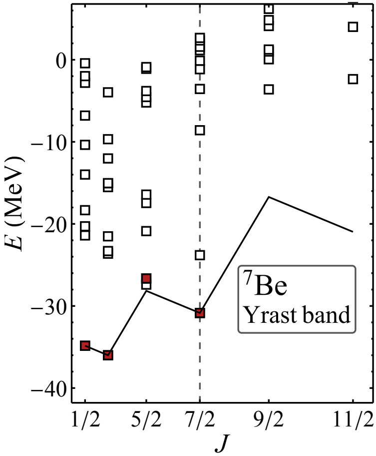
$$H = \alpha C_{\mathrm{Sp}(3,\mathbb{R})} + \varepsilon H_0 + \beta C_{\mathrm{SU}(3)} + a_L \mathbf{L}^2 + a_S \mathbf{S}^2 + \xi \mathbf{L}\cdot\mathbf{S}$$

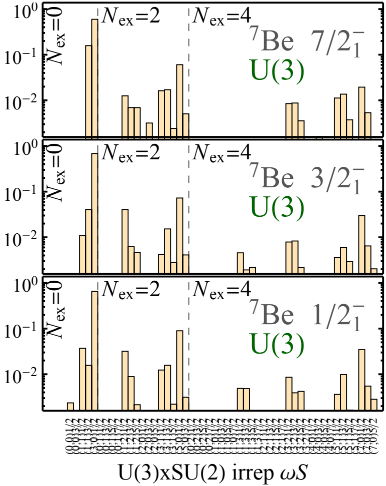
A. E. McCoy, M. A. Caprio, T. Dytrych, and P. J. Fasano, Phys. Rev. Lett. **125**, 102505 (2020).

Decomposition by U(3) content

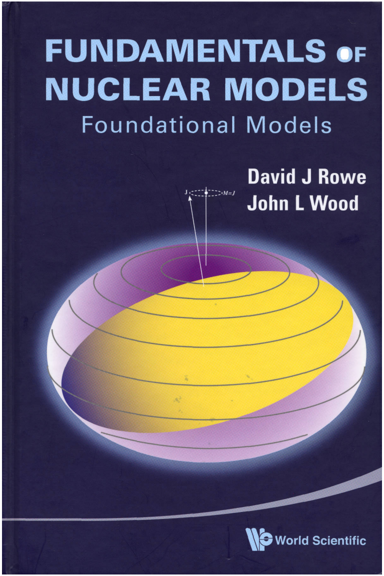
Yrast band up to maximal "valence" angular momentum has U(3)

$N_{\mathrm{ex}}(\lambda,\mu) = 0(3,0)$ $S = 1/2$



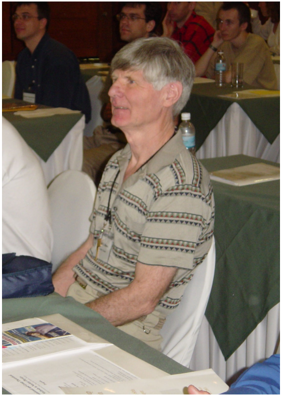


JISP16 (no Coulomb), SpNCCI, $N_{\sigma,\max} = N_{\max} = 6$, $\hbar\omega = 20\,\mathrm{MeV}$



Chapter 1

Elements of nuclear structure



David Rowe, Playa del Carmen, 2003.

Further reading

D. J. Rowe and J. L. Wood, *Fundamentals of Nuclear Models: Foundational Models* (World Scientific, Singapore, 2010).

http://www.worldscibooks.com/physics/6209.html

D. J. Rowe, *Nuclear Collective Motion: Models and Theory* (World Scientific, Singapore, 2018).

J. Suhonen, *From Nucleons to Nucleus* (Springer-Verlag, Berlin, 2007).

http://dx.doi.org/10.1007/978-3-540-48861-3

Richard F. Casten, *Nuclear Structure from a Simple Perspective*, 2ed (Oxford Univ. Press, 2000).

P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, New York, 1980).